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$SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$, the classical and quantum q-deformations of the SU(2) algebra: III. When q is a root of unity*

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Abstract. The q-oscillator of type one and the classical and the quantum q-deformations of SU(2) algebra realized through the q-oscillators are studied in the case of q being roots of unity, i.e. $q^k = 1$. The q-excitations are found to be non-bosonic. In particular, when k is of rank 4 and 6, the q-excitations are shown to be fermionic and parafermionic respectively.

1. Introduction

We pointed out [1-4]† that the q-deformation of SU(2) algebra can be realized at classical level in Poisson brackets, and it is denoted $SU_{q,\hbar\to 0}(2)$. This was made in classical system with q-deformed oscillator of two different types. Through canonical quantization, $SU_{q,\hbar\to 0}(2)$ is transferred to the conventional $SU_q(2)$ algebra expressed in Lie brackets, denoted $SU_{q,\hbar}(2)$. We emphasized that the $\hbar$-quantization and q-deformation are two independent concepts and this should be true for all quantum groups [5-9].

In this paper we continue the studies on the classical and quantum q-deformations of SU(2) algebra via the q-deformed oscillators when q is the root of unity [4]. Here q is called a root of unity of rank k, if $q^k = 1$, for k being any real number. We concentrate in this paper on the type one q-oscillator given in (I), (II) and [1]. Similar analysis can be made for the type two q-oscillators as well, but we omit it for brevity.

As we did in (I) and (II), we start by studying the classical dynamical system of q-oscillators, and show the q-deformed SU(2) algebra reduces to $SU_{q,\hbar\to 0}(2)$ when q is real or even rank root of unity. After quantization it becomes $SU_{q,\hbar}(2)$. One of the most interesting and important issues for the case of q being the roots of unity is that the excitations in the q-oscillators are found to be non-bosonic. Especially when the rank k is 4 and 6, the q-excitations are fermionic and parafermionic respectively.

This paper is organized in the following way. In section 2, we give a slightly modified approach to the realization of $SU_q(2)$ algebra via type one q-deformed oscillators with the parameter q being an arbitrary real number or root of unity. In section 3, we investigate the properties of the classical q-oscillator when q is a root of unity. Through

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† Here, and in what follows, we refer to [2, 3] as (I) and (II).

the canonical quantization we obtain the quantum q-oscillator. A well defined excitation number operator is introduced. A discussion on the direct product of the Fock spaces to produce the j-representation of the $SU_q(2)$ algebra is given. Section 4 is devoted to the excitations of the quantum q-oscillator, which are usually non-bosonic. A detailed discussion for $k=4$, 6 is provided, when the q-quanta are fermions and parafermions respectively. Their classical counterparts are indicated. In section 5, some brief discussions are presented.

2. q-deformations of SU(2) when q is a root of unity

Up to now, most of the investigations on the type one q-oscillator have been directed toward the case of q being real [9–11]. In the following, however, we give a more general discussion to include q being either real or roots of unity.

The algebra $SU_{q,\hbar\to 0}(2)$ can be realized in a classical system with two q-deformed oscillators with Hamiltonian

$$H' = z_1'\bar{z}_1' + z_2'\bar{z}_2'. \tag{2.1}$$

If one takes q to be 1, the variables z_i', $\bar{z}_i'$ reduce to undeformed variable z_i, $\bar{z}_i$, and H' to the Hamiltonian H for the undeformed system. For arbitrary q (i.e., q is real or the root of unity), the following relations between these two sets of variables hold in a modified form

$$z_i' = \sqrt{\frac{\sinh(\gamma\bar{z}_i z_i)}{\sqrt{\gamma \sinh\gamma}\,\bar{z}_i z_i}}\, z_i\, e^{i\gamma\alpha(z_i\bar{z}_i)} \qquad \bar{z}_i' = \left(\sqrt{\frac{\sinh(\gamma\bar{z}_i z_i)}{\sqrt{\gamma \sinh\gamma}\,\bar{z}_i z_i}}\right)^{*} \bar{z}_i\, e^{-i\gamma\alpha(z_i\bar{z}_i)} \tag{2.2}$$

(without summation over i), and $\gamma = \log q$. The complex conjugate is taken for the square root which may be taken of some negative value in the case of q being roots of unity.

For the mode z_i', we define the magnitude for the oscillation of q-oscillator, $\mathcal{N}_{iq} = \bar{z}_i' z_i'$. When $q=1$, one has $\mathcal{N}_i = \bar{z}_i z_i$, the magnitude for the undeformed oscillator. For arbitrary q, $\mathcal{N}_{i1}$ is related to $\mathcal{N}_i$ in the following way:

$$\mathcal{N}_{iq} = |[\mathcal{N}_i]|. \tag{2.3}$$

The mode sets for the undeformed and deformed oscillators can be constructed for modes z_i and z_i' respectively according to their magnitudes. Clearly, if $q=1$ the magnitude $\mathcal{N}_{i1} = \mathcal{N}_i$ takes the value from 0 to ∞, and so does it if q is an arbitrary real number, which is discussed in (I). When q is a root of unity, however, $\mathcal{N}_{iq}$ is finite, which is investigated in detail in the following sections.

In the phase space (V, Ω) of undeformed oscillators in (I) and (II), $\Omega = -i\sum_i (dz_i \wedge d\bar{z}_i)$, one has the Poisson brackets for the deformed variables,

$$\{\bar{z}_i', z_j'\} = \pm i\delta_{ij}\sqrt{\frac{\gamma}{\sinh\gamma}}\,|\cosh(\gamma\bar{z}_i z_i)|$$
$$\{z_i', z_j'\} = \{\bar{z}_i', \bar{z}_j'\} = 0. \tag{2.4}$$

It can be easily checked that there are always modes z_i' so that the observables $J_+ = z_i'\bar{z}_2'$, $J_- = z_2'\bar{z}_1'$ and $J_3 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2)$, satisfy the following relations in Poisson brackets:

$$\{J_3, J_\pm\} = (-i)(\pm J_\pm) \tag{2.5a}$$

$$\{J_+, J^-\} = (-i)\eta[2j_3] \tag{2.5b}$$

where $\eta=+$. The modes we want are those that satisfy

$$\eta=\operatorname{sign}(\tanh(2\gamma\mathcal{N}_1))=\operatorname{sign}(\tanh(2\gamma\mathcal{N}_2)). \tag{2.6}$$

When q is real, condition (2.6) holds for $\eta=+$, and hence the $\mathrm{SU}_{q,\hbar\to 0}(2)$ symmetry exists; while q is kth rank root of unity, part of the modes gives rise to $\eta=+$ and hence $\mathrm{SU}_{q,\hbar\to 0}(2)$†.

Based upon (2.4), one can perform the canonical quantization of this oscillator system [1]. As we remarked in (I) and (II), the deformed observables $J_\pm$, and J_3 are all defined on phase space (V,Ω). The canonical quantization is carried out by replacing the basic Poisson brackets by basic commutators for operators,

$$[a_i,a_j^\dagger]=\delta_{ij} \qquad [a_i^\dagger,a_j^\dagger]=[a_i,a_j]=0 \tag{2.7}$$

while the variables z_i', $\bar{z}_i'$ and $\mathcal{N}_{iq}$ are replaced by operators a_i', $a_i'^\dagger$, and $N_{iq}=a_i'^\dagger a_i'$ respectively. We choose normal ordering so as to have

$$[a_i',a_i'^\dagger]=\sqrt{\frac{\sinh\gamma}{\gamma}}\left(\left|\frac{[N_i+1]}{N_i+1}\right|-\left|\frac{[N_i]}{N_i}\right|\right). \tag{2.8}$$

And the quantum counterpart of (2.2) is‡

$$\begin{aligned} a_i'&=a_i\sqrt{\frac{[N_i]}{\sqrt{\gamma\sinh\gamma}N_i}}=\sqrt{\frac{[N_i+1]}{\sqrt{\gamma\sinh\gamma}(N_i+1)}}\,a_i \\ a_i'^\dagger&=\left(\sqrt{\frac{[N_i]}{\sqrt{\gamma\sinh\gamma}N_i}}\right)^* a_i^\dagger=a_i^\dagger\left(\sqrt{\frac{[N_i+1]}{\sqrt{\gamma\sinh\gamma}(N_i+1)}}\right)^* \end{aligned} \tag{2.9}$$

where $[N_i]$ may be negative, when q is root of unity, but the excitation number operator N_{iq} is always non-negative.

The quantum observables are $J_+=a_1'^\dagger a_2'$, $J_-=a_2'^\dagger a_1'$, $J_3=(a_1^\dagger a_1-a^\dagger-a_2^\dagger a_2)$. From the basic commutators, we have the algebra satisfied by the operators

$$[J_3,J_\pm]=\pm J_\pm \tag{2.10a}$$

$$[J_+,J_-]=\frac{\sinh\gamma}{\gamma}(|[N_1+1][N_2]|-|[N_2+1][N_1]|). \tag{2.10b}$$

We can always select the states that obey the condition

$$\eta=\operatorname{sign}([N_1][N_2+1])=\operatorname{sign}([N_1+1][N_2]). \tag{2.11}$$

In fact, the $\eta=+$ case supplies the $\mathrm{SU}_{q,\hbar}(2)$ symmetry. For q real we have $\eta=+$ case only, and hence $\mathrm{SU}_{q,\hbar}(2)$ symmetry only; for q being even rank root of unity, $\eta=+$ or $-$; for q being odd rank root of unity, more possibilities arises§.

In the previous investigations, as q is taken to be real, η is always $+$. We are to look into the case where q is root of unity of even rank, η may be $+$ or $-$. As can be seen clearly in the following section, while the parameter q is root of unity of even rank, the Fock space of the q-oscillator splits into infinite number of subspaces denoted as $V_{k^i}^{L_i}$ (see next section), which is finite dimensional. In each subspace, $[N_i]$ is always positive or negative. To get the representation space of $J_\pm$ and J_3, one may make

† Part of the modes gives rise to $\eta=-$ and hence $\mathrm{SU}_{q,\hbar\to 0}(1,1)$; see section 5.

‡ For the following discussion the phases $\alpha(z_i\bar{z}_i)$ will be irrelevant; we neglect them.

§ When η is $-$, $\mathrm{SU}_{q,\hbar}(1,1)$ symmetry arises; see the short discussion provided in section 5.

direct-product of the subspaces of the two oscillators, and the parameter l_1, l_2 for the two components are both even or odd, one has η to be $+$, and otherwise $-$.

3. q-oscillator when $q^k = 1$

In this section, we investigate the properties of modes, magnitude and mode spaces for the classical q-oscillator when q is root of unity of even rank, and perform the canonical quantization to obtain the operators, and Fock space of the quantum q-oscillator. We give a more reasonable particle number operator for the q-quanta. A discussion regarding direct product of the Fock spaces to produce the representation of the $\mathrm{SU}_q(2)$ algebra. As we concentrate on a single q-oscillator, we neglect the index i $(=1, 2)$.

First, let us observe a q-deformed oscillator in classical mechanics when q is root of unity of even ranks. Let $\gamma = 2\pi\mathrm{i}/k = \pi\mathrm{i}/p$, where $p = k/2$ is an integer and $q = \mathrm{e}^{2\pi\mathrm{i}/k} = \mathrm{e}^{\pi\mathrm{i}/p}$. Hence the magnitude $\mathcal{N}_q$ has maximum and minimum in its absolute value:

$$0 \leqslant \mathcal{N}_1 = |[\mathcal{N}]| = \left|\frac{\sin(\pi\mathcal{N}/p)}{\sin(\pi/p)}\right| \leqslant \frac{1}{\sin(\pi/p)} \tag{3.1}$$

the rightmost equality stands at $\mathcal{N} = \frac{1}{2}pl$ and $l = 1, 3, 5, \ldots$. In other words, the magnitude $\mathcal{N}$ is (continuous) and finite, regardless of the magnitude of the undeformed ordinary oscillator that may be infinitely large. The leftmost equality stands for $\mathcal{N} = pl$ and $l = 0, 1, 2, \ldots$, indicating that the magnitude of the q-deformed oscillator may be 0 while that of the undeformed oscillator is not 0. In the following, one can see that this property of the q-magnitude is the counterpart of quantum q-condensation of bosonic quanta.

For generic $\mathcal{N}$ in the interval $[pl, pl+p)$, $\mathcal{N}_q$ experiences an increasing from 0, and arriving a maximum value $(\sin(\pi/p))^{-1}$, and then going down to 0 again. This property of the classical q-magnitude may be called 'q-saturation'. And the modes with the magnitude in this interval completes a subset of oscillation modes, named $\tilde{V}_p^l$, and it is obvious to see that

$$\tilde{V} = \bigcup_{l=0}^{\infty} \tilde{V}_p^l \tag{3.2}$$

where $\tilde{V}$ is the set of oscillation modes for the ordinary oscillator.

To understand this saturation property better, we consider $p = 2$ as an example. In this situation, one has $\mathcal{N}_q \geqslant 1$; when $\mathcal{N} = 2l+1$, $l = 0, 1, 2, \ldots$ one has maximum value 1. If $\mathcal{N}$ has the spectrum in the interval $[2l, 2l+2)$, mode z' of the magnitude $\mathcal{N}_q$ completes a mode set $\tilde{V}_2^l$. As $\mathcal{N}$ goes up starting at $2l$, $\mathcal{N}_q$ becomes greater and greater, and at $\mathcal{N} = 2l+1$, $\mathcal{N}_q$ gains its maximum value 1. When $\mathcal{N}$ continues to increase, however, $\mathcal{N}_q$ begins decreasing, to 0 at $\mathcal{N} = 2l+2$. In the next section, we will see that the saturation property in this situation ($p = 2$) is just the classical counterpart of the Pauli exclusion principle.

As in usual cases, after the canonical quantization the mode set gives rise to the Fock space for the quantum q-oscillator, and magnitude $\mathcal{N}_q$ the particle number operator $N_q = a'^{\dagger}a'$, where a' and $a'^{\dagger}$ are annihilation and creation operators respectively. When the eigenvalues of N, which are integers in the interval $[pl, pl+p)$, the eigenvalue of N_q increases at a series of discrete values, gains maxima at one or two points, and then goes back to 0. The cycle completed is a Fock subspace V_p^l.

In fact, the Fock subspace can be constructed with the particle number eigenstates of undeformed oscillator. When q is a root of unity of even rank, the complete Fock subspace V_p^l can be constructed

$$V_p^l = \{|pl\rangle, |pl+1\rangle, \ldots, |p(l+1)-1\rangle\}$$
$$= \left\{ \left| pl+n \right\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}} \left| pl \right\rangle, n = 0, 1, 2, \ldots, p-1 \right\} \tag{3.3}$$

where $l = 0, 1, 2, \ldots$. It is clear that V_p^l are identical as l takes different values, and

$$V = \bigcup_{l=0}^{\infty} V_p^l \tag{3.4}$$

where V is the Fock space for the undeformed oscillator which is infinite dimensional. In the Fock subspace V_p^l, there is vacuum state $|pl\rangle$. But only the vacuum state $|0\rangle$ in the Fock subspace V_p^0 is simultaneously the vacuum state for the undeformed oscillator. It should be emphasized that the operator N_q describes the vacuum states in various Fock subspaces correctly:

$$N_q|pl\rangle = 0 \tag{3.5}$$

but the eigenvalues of this operator are not always integers, so it is improper to be selected as particle number operator. This is a similar situation to that in parastatistics, where the eigenvalues of $a^\dagger a$ are not integers, not even rational numbers, and a new, well defined number operator should be given.

It is easy to see that we can actually define an operator $\tilde{N}_q$ which has the eigenvalue $\tilde{N}_q \equiv N(\mathrm{mod}\, p)$, and we write formally in operators $\tilde{N}_q \equiv N(\mathrm{mod}\, p)$, to have a proper spectrum $0, 1, 2, \ldots, p-1$ that are integers. And this new operator describes vacuum states correctly,

$$\tilde{N}_q|pl\rangle = 0. \tag{3.6}$$

In the following section, when we analyse the statistical properties of the q-quanta, we will find the definition of the new particle number operator more reasonable.

Finally in this section, we make some clarification regarding the role played by the η sign. As one can notice, $[N] \leqslant 0$ in subspace V_p^l with even l, $[N] \geqslant 0$, is that with odd l. And so when one makes direct multiplication of the Fock subspaces of the two oscillators, $V_{p_i}^{l_i}$, to obtain the representation space of the algebra $\mathrm{SU}_{q,\hbar}(2)$† one should notice the following two cases to fix the sign η: when $\Delta l = l_1 - l_2 \equiv 0$, η is $+$; and if $\Delta l \equiv 1$, η is $-$.

4. Non-bosonic q-quanta and their symmetries

In the present section, we concentrate on the statistical properties of q-quanta, especially the $k = 4, 6$ cases which are the most physically appealing. We state that the operator $\tilde{N}_q$ coincides with the conventional particle number operator [14].

† Or $\mathrm{SU}_{q,\hbar}(1,1)$; see section 5.

If $p=2$, the set of modes for the classical q-oscillator is $\tilde{V}_2^l=\{z', \mathcal{N}\in[2l, 2l+2)\}$. After canonical quantization, one obtains the complete and irreducible Fock spaces $V_2^l=\{|2l\rangle, |2l+1\rangle\}$. It can be checked that the operators for the q-oscillator are fermionic operators satisfying the following algebraic relations

$$[a', a'^{\dagger}]_+=1 \qquad a'^2=a'^{\dagger 2}=0. \tag{4.1}$$

When they act on Fock space

$$\begin{aligned} &a'|2l\rangle=0 \qquad a'|2l+1\rangle=\sqrt{(-1)^l}|2l\rangle \\ &a'|2l\rangle=(\sqrt{(-1)^l})^*|2l+1\rangle \qquad a'^{\dagger}|2l+1\rangle=0. \end{aligned} \tag{4.2}$$

The newly defined particle number operator reads

$$\tilde{N}_q=N_q=(a'^{\dagger}a'-a'a'^{\dagger})/2+\tfrac{1}{2} \tag{4.3}$$

and the spectrum is $\{0, 1\}$. This is the traditional definition for the fermion [14].

The saturation property of the magnitude $\mathcal{N}_q$ is just the classical counterpart of the Pauli exclusion principle [4]. As $\mathcal{N}$ increases, $\mathcal{N}_q$ experiences an increasing from 0 to 1 and then a decreasing from 1 to 0. After canonical quantization, the eigenvalue of $\tilde{N}_q$ jumps from 0 to 1. The occupation number cannot be 2 or greater, i.e. quanta are excluded by the state occupied by a single quantum. This is what is stated by the Pauli exclusion principle. It is of interest to notice that the SU(2) algebra is now realized by two identical bosonic oscillators and two identical fermionic oscillators in interaction.

For the case of $p=3$, the magnitude $\mathcal{N}_q$ is between 0 and $2/\sqrt{3}$. After the canonical quantization, the spectrum of N_q in the complete irreducible Fock space $V_3^l=\{|3l\rangle, |3l+1\rangle, |3l+2\rangle\}$ is $\{0, 1, 2\}$, which is exactly the spectrum for the parafermion [14]. And the operators a' and $a'^{\dagger}$ obey the following algebraic relations

$$a'^{\dagger}a'a'+a'a'a'^{\dagger}=a' \qquad a'a'^{\dagger}a'=a' \qquad a'^3=a'^{\dagger 3}=0 \tag{4.4}$$

which is just the $\beta=1$ case in [16]. According to [15], equations (4.4) are the algebraic relations for parafermions, except that a factor $\sqrt{2}$ is to be multiplied to a' and $a'^{\dagger}$ to recover the conventional algebraic relation. And therefore, the good particle number operator should be the following [14]:

$$a'^{\dagger}a'-a'a'^{\dagger}+\frac{p-1}{2} \tag{4.5}$$

and this is exactly $\tilde{N}_q$, with $p-1=2$ the order of parafermions [14].

5. Remarks and discussion

In the above, we analysed the realization of $SU_q(2)$ algebra, and the non-bosonic excitations, when q is root of unity of even rank in type one model. In fact, there exist non-bosonic excitations in the q-oscillators of type one, when q is root of unity of odd rank or fractional rank, or even irrational rank; however, this has nothing to do with the realization of $SU_q(2)$ algebra, and so will be discussed in a separate paper [12]. Similar analysis apply to the type two q-oscillator.

It is also interesting to note that the $SU_q(1,1)$ symmetry may be found in the system of q-oscillators in the case of q being root of unity. In fact if (2.6) holds for $n=-$, (2.5) stand for $SU_{q\hbar\to 0}(1,1)$, the classical counterpart of the q-deformed $SU_q(1,1)$ algebra [17]. When canonically quantized, $SU_{q\hbar}(1,1)$ algebra [17] arises with (2.11) holds for $\eta = -$.

Another thing worth noting is the multi-deformation of harmonic oscillators, and the multi-deformed algebras set up in (II). We got a chain for the complex variables of the q-deformed oscillators. In fact, that chain is endless if every deformation parameter is real. If the deformation parameters can be roots of unity, however, the chain may be truncated. The further deformations of the q-oscillator may be trivial. For example, when q_1 is root of unity of the fourth rank, then the q_1-oscillator is fermionic. It can be checked easily that the q_2-deformation is trivial. Because for a fermionic oscillator, (4.1) are valid, so we have $N_q^2 = a'^\dagger a' a'^\dagger a' = a'^\dagger(1 - a'^\dagger a')a' = N_q$, and then

$$\sinh(\gamma N_q) = \sum_{i=0}^{\infty}\left((-)^i \frac{N_q^{2i+1}}{(2i+1)!}\right) = (\sinh\gamma)N_q \tag{5.1}$$

and this means $[N_q] = N_q$. For parafermion case in type one model where $k = 6$, from (4.4), one can verify that $N_q^2 = a'^\dagger(a'a'^\dagger a') = a'^\dagger a' = N_q$, and then $[N_q] = N_q$. So we remark that there is no realization of q-deformed algebras by fermionic or parafermionic oscillators except trivial ones.

In [9], however, a q-deformation of Clifford algebra is put forward to realize the q-deformations of Lie algebras B_n and D_n. In fact, the q-Clifford algebra coincides with Clifford algebra, so the deformed B_n and D_n algebras seem to be trivial. We shall explain this subject in detail elsewhere.

Finally, we would like to point out that the relationship between the j-representation of the $SU_q(2)$ algebra and the Fock spaces of the q-oscillators is an interesting topic still in progress.

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